Name:

ID:

March 16th, 2015

Winter quarter, Version A

Final Exam

Instructions: Do not open this exam until told to do so. Once the test starts, briefly look over the exam so that you can pace yourself. **No calculators are permitted or necessary**. Show your work for each problem and clearly indicate your answers. Crossed out or erased work will not be graded.

Please read and sign the academic honesty statement below.

I certify that this exam was taken by the person named and done without any form of assistance including calculator, cell phone, books, notes and other people.

Signature:

GOOD LUCK!

Scores:

1	2	3	4	5	6	7	8	9	10	11	12	13	Σ

Total is 111 points.

Formulas

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
C	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
$\vdots$	$\vdots$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
e^{-at}	$\frac{1}{s+a}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}

1-SP:	$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$
2-SP:	$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$	$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$	$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$
	$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$	$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

1. [6 points] Solve the following differential equation using the power series method:

$$y^{''} - 9y = 0$$

If possible, write your answer in terms of elementary functions (that is, no longer writing it as a power series).

2. [10 points] Solve the following differential equation using the power series method.

$$(x^2 + 1)y'' - 4xy' + 6y = 0$$

3. [4 points] Use power series to verify that

$$\frac{d}{dx}e^{-2x} = -2e^{-2x}$$

4. [5 points] Consider the autonomous equation

$$x^{'}(t) = -3e^{x}\cos(\pi x)$$

(a) Draw the phase diagram. Make sure your phase diagram includes at least the interval $-2 \leq t \leq 2$.

(b) If $x(0) = \frac{1}{2}$, what is the value of $\lim_{t\to\infty} x(t)$?

5. [6 points] Consider the system:

$$\begin{aligned} x' &= 2x + \pi y - z \\ y' &= 2y - z \\ z' &= 2z \end{aligned}$$

(a) Write the system as a single equation using vectors and a matrix.

(b) Why can this system **not** be solved using eigenvalues?

(c) Solve the system using matrix exponentiation.

6. [4 points each] Show each of the following:

(a) Suppose that $g(t)$ is a differentiable function of exponential order. For simplicity, assume that $\lim_{t\to\infty} e^{-st}g(t) = 0$. Verify that:

$$\mathcal{L}\left\{g^{''}(t)\right\} = s^2G(s) - s\,g(0) - g^{'}(0)$$

(b) Suppose that $f(t)$ is a function of exponential order. Verify that

$$\mathcal{L}\left\{-t\,f(t)\right\} = F^{'}(s)$$

7. Solve each of the following ODEs.

(a) [6 points] $x^{''} - 6x^{'} + 9x = -2\cos(3t)$

(b) [3 points] $x^{''} - 6x^{'} + 9x = e^{3t}$

8. [6 points] Consider the following function:

$$f(t) = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } 2 \leq t < 4 \\ t & \text{if } t \geq 4 \end{cases}$$

(a) Write $f(t)$ as a single function using Heaviside functions, so that it is not defined piecewise.

(b) Find $\mathcal{L}\{f(t)\}$ using any method that you have learned.

9. [4 points each] For each of the following lists of functions, check for linear dependence or independence using any method that we have learned:

   (a) $\sinh(x)$, $e^x, 2e^{-x}$

   (b) $1, x, x^2, x^3$

10. [5 points each] Solve the following ODEs. If the problem says to give an implicit solution, you do not need to solve for y. Otherwise, solve all the way until your answer is of the form $y = f(x)$, where $f(x)$ is some expression in the variable x.

(a)

$$y^{'} + \frac{y}{x} = e^{x^2}$$

(b)

$$y^{'} = 2xy + 3x - 2y - 3$$

(c) Give an implicit solution:

$$y' = \frac{3xy}{x^2 + 1}$$

(d)

$$y' + \sin(x)y = \sin(x)y^4$$

(e)

$$y^{(3)} - y^{(1)} = 0$$

11. Use the Laplace transform to solve the following ODEs.

    (a) [5 points] Assuming the initial conditions $x(0) = x^{'}(0) = 0$, solve:

$$x^{''} = t^2 u(t-2)$$

(b) [5 points] Assuming the initial condition $x(0) = 0$ and $x^{'}(0) = 1$, solve:

$$x^{''} + 4x = \delta(t)$$

(c) [6 points] Assuming the initial conditions $x(0) = x^{'}(0) = 0$, solve:

$$x^{''} + x = 2t$$

Use a convolution when computing the inverse Laplace transform in this problem.

12. [8 points] Find a general **real-valued** solution to the following system:

$$x' = 5x - y$$

$$y' = 4x + 5y$$

13. [? points] Draw your favorite animal.